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Back to the alg. of differential operators $D(G)$

$\mathfrak{g} = \mathrm{Lie}(G)$

$\mathcal{O}(G) \otimes \mathfrak{g} \simeq \Theta_G$.

. $\mathcal{O}(G) \otimes U(\mathfrak{g}) \xhookrightarrow{L} D(G)$

$$f \otimes x \longmapsto f x_L, \quad x \in \mathfrak{g}$$

. We proved that L is an isomorphism: $\mathcal{O}(G) \otimes U(\mathfrak{g}) \simeq D(G)$

. Let $D_\ell(G)$ be the alg of left-invariant diff. operators: α:

$\forall f \in \mathcal{O}(G)$, $\forall g \in G$, $\lambda_g \cdot (\alpha f) = \alpha(\lambda_g f)$

where: $(\lambda_g f)(y) = f(g^{-1}y)$ $\quad \forall f \in \mathcal{O}(G)$, $\forall g, y \in G$.

. $U(\mathfrak{g}) \subset D_\ell(G)$

. G acts on $\mathcal{O}(G) \otimes U(\mathfrak{g})$ by acting on $\mathcal{O}(G)$ by λ_g, $g \in G$, and trivially on $U(\mathfrak{g})$

. this induces a G-action on $D(G)$

$$D(G)^G \simeq (\mathcal{O}(G) \otimes U(\mathfrak{g}))^G \simeq \mathcal{O}(G)^G \otimes U(\mathfrak{g}) \simeq U(\mathfrak{g}) \subset D_\ell(G)$$

. $D_\ell(G) \subset D(G)^G$?!

$\alpha = \sum_i h_i \otimes u^i \in D_\ell(G)$, $\forall f \in \mathcal{O}(G)$, $\forall y \in G$

$\lambda_g(\alpha f)(y) = \sum_i \underline{h_i(g^{-1}y)} (u^i_L f)(g^{-1}y)$

$\alpha(\lambda_g f)(y) = \sum_i \underline{h_i(y)} (u^i_L f)(g^{-1}y)$ $\quad$ since $u^i \in D_\ell(G)$

concl: $\underline{D_\ell(G) \simeq D(G)^G \simeq U(\mathfrak{g})}$

Prop (weight filtration)

(i) $T G_p V \subset G_p V$

(ii) $(G_p V)_{(n)} (G_q V) \subset G_{p+q} V \qquad n \in \mathbb{Z}$

(iii) $(G_p V)_{(n)} (G_q V) \subset G_{p+q-1} V \qquad n \in \mathbb{Z}_{\geq 0}$.

proof: (i) $[T, a^i_{(-n)}] = n\, a^i_{(-n-1)} \qquad i \in I,\ n \in \mathbb{Z},\ T|0\rangle = 0$

(ii) $n<0$: we do exactly as for $(F^p V)_p$ for $n<0$ (Borcherds identity + induction)

$\underline{n \geq 0}$ $\quad G_{p+q-1} V \subset G_{p+q} V$, it suffices to prove (iii)

$a \in G_p V,\ b \in G_q V$ homogeneous, $n \geq 0 \qquad a_{(n)} b \in G_{p+q-1} V$

Recall: $(V_\Delta)_{(n)} (V_{\Delta'}) \subset V_{\Delta+\Delta'-n-1} \qquad [H, a_{(n)}] = -(n+1) a_{(n)} + (Ha)_{(n)}$

therefore (ii) with $n \geq 0$ and (iii) will be a consequence of the lemma:

Lemma 2: $F^p V_\Delta = G_{\Delta-p} V_\Delta \qquad G_p V_\Delta = G_p V \cap V_\Delta,\ F^p V_\Delta = F^p V \cap V_\Delta$.

| therefore: $\mathrm{gr}^F V \simeq \mathrm{gr}_G V$.

lemma 2 $\Rightarrow$ $a_{(n)} b \in (F^{\Delta_a - p} V_{\Delta_a})_{(n)} F^{\Delta_b - q} V_{\Delta_b} \underset{\substack{\uparrow \\ \text{statement for } F^\bullet V}}{\subset} F^{\Delta_a - p + \Delta_b - q - n} V_{\Delta_a + \Delta_b - n - 1}$

$\underset{\text{lemma 2}}{=} G_{p+q-1} V_{\Delta_a + \Delta_b - n - 1}$ ✓

proof of the lemma 2: induction on p, $p \leq 0$ $\quad F^p V_\Delta = V_\Delta$

$G_{\Delta - p} V_\Delta \supset G_\Delta V_\Delta = V_\Delta$. Conversely if $a \in V_\Delta$, $a = a_{(-1)}|0\rangle \in G_\Delta V_\Delta$. ✓

$\underline{p>0}$

By lemma 1, $F^p V_\Delta$ is generated by elements $v = a_{(-i-1)} b$, $a \in V_{\Delta_a}$, $b \in F^{p-i} V_{\Delta_b}$ s.t. $\Delta_a + \Delta_b + i = \Delta$.

It suffices to show that $a_{(-i-1)} b \in G_{\Delta-p} V_\Delta$.

$$v = a_{(-i-1)} b \underset{p}{\in} G_{\overset{\Delta-p}{\overset{\|}{\Delta_a+\Delta_b-p+i}}} V_{\underbrace{\Delta_a+\Delta_b+i}_{=\Delta}} = G_{\Delta_p} V_\Delta \checkmark$$

$a \in G_{\Delta_a} V_{\Delta_a}$ $b \in G_{\Delta_b - p + i} V_{\Delta_b}$ by induction hypothesis.

prop (ii) with $n = -i-1 < 0$

$\underline{\text{Concl}}$: $F^p V_\Delta \subset G_{\Delta-p} V_\Delta$.

<u>It remains to show the other inclusion</u>: $G_p V_\Delta \subset F^{\Delta-p} V_\Delta$.

Induction on the length of monomials of the form: $v = a^{i_1}_{(-n_1-1)} \cdots a^{i_r}_{(-n_r-1)} |0\rangle$

$\underline{r=0}$ $|0\rangle \in G_{\Delta-p} V_\Delta$ $\Delta = 0$ $G_{\Delta-p} V_\Delta \neq \{0\}$ only if $p=0$.

$r>0$: $v = a^{i_1}_{(-n_1-1)} w$ $w = a^{i_2}_{(-n_2-1)} \cdots |0\rangle$ $n_j \geq 0$ $\sum_{i=1}^{r} \Delta_{a^i} \leq p$, $\Delta_{a^{i_1}} + \Delta_w + n_1 = \Delta$

$$w \in G_{p-\Delta_{a^{i_1}}} V_{\Delta_w} \underset{\substack{\text{induction} \\ w \text{ has length } r-1}}{\subset} F^{\Delta_{a^{i_1}} + \Delta_w - p} V_{\Delta_w}$$

$$v = a^{i_1}_{(-n_1-1)} w \in F^{\Delta_{a^{i_1}} + \Delta_w - p + n_1} V_{\underbrace{\Delta_{a^{i_1}} + \Delta_w + n_1}_{=\Delta}} = F^{\Delta-p} V_\Delta. \qquad \boxed{\text{km}}$$

$\underline{\mathrm{gr}_G V}$ comm. VA.

$\mathrm{gr}^F V = \bigoplus_{p \geq 0} F^p V / F^{p+1} V$ $\mathrm{gr}_G V = \bigoplus_{p \geq 0} G_p V / G_{p-1} V$ $G_{-1} V = \{0\}$

$$\mathrm{gr}^F V = \bigoplus_{\underline{\Delta \geq 0}} (\mathrm{gr}^F V)_\Delta = \frac{G_{\Delta-p} V_\Delta}{G_{\Delta-p-1} V_\Delta} = \mathrm{gr}_G V$$

§7. The lisse condition

Simplest case: when X_V has dimension 0.

(We assume: R_V is finitely generated $\Leftrightarrow$ V is finitely strongly generated)

Let $\{a^i : i \in I\}$ be a set of vectors of a good vertex algebra.

$\{a^i : i \in I\}$ are strong generators of V $\Leftrightarrow$ the image of $\{a^i : i \in I\}$ in R_V generate R_V.

Rem: $X = \operatorname{Spec} R$ affine scheme of finite type $/\mathbb{C}$.

$\dim X = 0 \Leftrightarrow R$ is finite-dimensional $\mathbb{C}$-alg.

Def: We say that V is lisse, or C_2-cofinite, if $\boxed{\dim X_V = 0}$

($\Leftrightarrow \dim R_V < \infty$)

$R_V = V/F^1 V = V/C_2 V$

$\Leftrightarrow \dim SS(V) = 0$, $SS(V) = \operatorname{Spec} \operatorname{gr}^F V$ (Beilinson–Feigin–Mazur for Virasoro)

($\dim X_V = 0 \Rightarrow \dim (X_V)_{red} = 0 = \dim J_\infty (X_V)_{red} = 0 \Rightarrow \dim J_\infty X_V = 0$, $J_\infty (X_V)_{red} \overset{\text{top}}{=} J_\infty X_V$

$\mathbb{C}[J_\infty X_V] \twoheadrightarrow \operatorname{gr}^F V \Rightarrow SS(V) \subset J_\infty X_V$: $\pi_{\infty,0} : J_\infty X_V \to X_V$

If $\dim J_\infty X_V = 0$, then $\dim SS(V) = 0$

Conversely, if $\dim SS(V) = 0$ then $\pi_{\infty,0}(SS(V)) = X_V$ has dimension 0)

3 aspects to discuss

* Properties of lisse vertex algebras.
* Examples of lisse vertex algebras
* Why lisse vertex algebras are nice?

Examples and non-examples of lisse vertex algebras

- $X_{V^k(\mathfrak{g})} = \mathfrak{g}^*$ ⇒ $V^k(\mathfrak{g})$ is not lisse. Rem: does NOT depend on k

  $X_{Vir^c} = \mathbb{A}^1$ $\quad Vir^c$ not lisse. Rem: does not depend on c.

  $X_{\mathcal{D}^{ch}_{G,k}} \simeq T^*G$. $\quad$ NOT lisse as well.

To have examples of lisse VA, we need to consider simple quotient.

We will see: $X_{L_k(\mathfrak{g})} = \{0\}$ iff $k \in \mathbb{Z}_{\geq 0}$, $\mathfrak{g}$ simple Lie algebra.

Digression on associated varieties of primitive ideals of $U(\mathfrak{g})$

$\mathfrak{g}$ simple

$\mathfrak{g} = \underset{\underset{\alpha\in\Delta_+}{\bigoplus}\mathfrak{g}_{-\alpha}}{\underset{\parallel}{n_-}} \oplus \mathfrak{h} \oplus \underset{\underset{\alpha\in\Delta_+}{\bigoplus}\mathfrak{g}_{\alpha}}{\underset{\parallel}{n_+}}$ triangular decomposition, $\Delta = \Delta(\mathfrak{g},\mathfrak{h})$, Δ_+

$\mathfrak{h}$ Cartan subalgebra of $\mathfrak{g}$.

<u>EX</u>: $\mathfrak{g} = \mathfrak{sl}_n = \{x \in \mathrm{Mat}_n(\mathbb{C}) : \mathrm{tr}\, x = 0\}$ $\quad [x,y] = xy - yx$.

We can choose for $\mathfrak{h} = \left\{\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\} \subset \mathfrak{sl}_n$, $n_+ = \left\{\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}\right\} \subset \mathfrak{sl}_n$, $n_- = \left\{\begin{pmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{pmatrix}\right\} \subset \mathfrak{sl}_n$

$L_{\mathfrak{g}}(\lambda)$: irreducible representation of $\mathfrak{g}$ with <u>highest weight</u> $\lambda \in \mathfrak{h}^*$

$L_{\mathfrak{g}}(\lambda)$ is gen as $\mathfrak{g}$-module by some $v_\lambda \in \mathfrak{g}$ s.t. $n_+ . v_\lambda = 0$

$h . v_\lambda = \lambda(h) v_\lambda \quad \forall h \in \mathfrak{h}^*$

$\mu \leq \lambda$ iff $\lambda - \mu \in \bigoplus_{n\in\mathbb{N}} \mathbb{N}\alpha$

$(L_{\mathfrak{g}}(\lambda))_\mu = \{m \in L_{\mathfrak{g}}(\lambda) : h.m = \lambda(h) m \quad \forall h \in \mathfrak{h}\}$. ($\mu$ weight of $L_{\mathfrak{g}}(\lambda)$ if $L_{\mathfrak{g}}(\lambda)_\mu \neq \{0\}$).

$\dim \underset{\underset{\underline{\mathbb{C} v_\lambda}}{\parallel}}{L_{\mathfrak{g}}(\lambda)_\lambda} = 1$

<u>Fact</u>: $\dim L_{\mathfrak{g}}(\lambda) < \infty \iff \lambda \in P^+ = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \quad \forall \alpha \in \Delta_+\}$

$P = \bigoplus_{i=1}^{\ell} \mathbb{Z}\varpi_i \qquad \Delta \supset \Pi = \{\alpha_1 \ldots \alpha_\ell\} \quad \ell = \mathrm{rg}(\mathfrak{g}) = \dim \mathfrak{h}$.

$\{\varpi_1 \ldots \varpi_\ell\}$ dual basis of $\{\alpha_1^\vee, \ldots \alpha_\ell^\vee\}$.

Center $Z(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$.

$\forall h \in \mathfrak{h}$, $\forall z \in Z(\mathfrak{g})$ $\quad h.(z.v_\lambda) = z(h v_\lambda) = \lambda(h)\, z v_\lambda$. $\Rightarrow z v_\lambda \in L_\mathfrak{g}(\lambda)_\lambda$

$\dim L_\mathfrak{g}(\lambda)_\lambda = 1$: $\quad z v_\lambda = \chi_\lambda(z) v_\lambda$ for some $\chi_\lambda(z) \in \mathbb{C}$.

Moreover, $\forall u \in U(\mathfrak{g})$ $\quad z(u v_\lambda) = u(z v_\lambda) = \chi_\lambda(z)\, u v_\lambda$

Concl : $z \in Z(\mathfrak{g})$ acts via the scalar $\chi_\lambda(z)$.

$\chi_\lambda : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$, $z \longmapsto \chi_\lambda(z)$ is an alg. morphism.

central character of $L_\mathfrak{g}(\lambda)$ $\longrightarrow$ $\chi_\lambda \in \mathrm{Hom}_{Alg}(\underbrace{Z(\mathfrak{g})}_{\text{comm alg.}}, \mathbb{C})$ $\quad \chi_\lambda \in \mathrm{Specm}\, Z(\mathfrak{g})$

Recall : $\mathrm{gr}\, U(\mathfrak{g}) = S(\mathfrak{g})$ $\quad \underset{\simeq Z(\mathfrak{g})}{\mathrm{gr}\, Z(\mathfrak{g})} \simeq S(\mathfrak{g})^{\mathfrak{g}} \simeq \mathbb{C}[p_1 \ldots p_\ell]$

. Let J be a left ideal of $U(\mathfrak{g})$, then $\mathrm{gr}\, J = \bigoplus_{i \geq 0} J \cap U_i(\mathfrak{g}) / J \cap U_{i-1}(\mathfrak{g})$ is an ideal of $S(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$

$\mathcal{V}(J) := \{ x \in \mathfrak{g}^* : f(x) = 0 \quad \forall f \in \mathrm{gr}\, J \}$

zero locus of $\mathrm{gr}\, J$ in $\mathfrak{g}^*$ ($\cong \mathfrak{g}$ through the Killing form)

Set $\underset{\substack{\| \\ J_\lambda}}{\mathrm{Ann}_{U(\mathfrak{g})}}\, L_\mathfrak{g}(\lambda) = \{ x \in U(\mathfrak{g}) : x.m = 0 \quad \forall m \in L_\mathfrak{g}(\lambda) \}$

left-ideal of $U(\mathfrak{g})$. One can consider $\mathcal{V}(J_\lambda)$: associated variety of $L_\mathfrak{g}(\lambda)$.

More generally, an ideal J of $U(\mathfrak{g})$ is called <u>primitive</u> if J is the annihilator of some simple left-module of $U(\mathfrak{g})$ $\quad$ (ex : J_λ)

$\mathrm{Prim}\, U(\mathfrak{g}) = \{$ primitive ideals of $U(\mathfrak{g}) \}$ plays the role of $\mathrm{Spec}\, A$ for a comm $\mathbb{C}$-alg A.

two important things about primitive ideals of $U(\mathfrak{g})$

1) Duflo: if $J \in \mathrm{Prim}\, U(\mathfrak{g})$, then J is the annihilator of some $L_{\mathfrak{g}}(\lambda)$, $\lambda \in \mathfrak{h}^*$

" $\mathrm{Ann}_{U(\mathfrak{g})} M$.

2) Joseph: $\mathcal{V}(J_\lambda) = \mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})} L_{\mathfrak{g}}(\lambda)) \subset \mathcal{N}$ (not difficult) $(\mathfrak{g}^* \simeq \mathfrak{g})$

Moreover: $\mathcal{V}(J_\lambda)$ is irreducible.

since $\mathcal{N}$ is a finite union of G-orbits: $\mathcal{V}(J_\lambda) = \overline{\mathbb{O}}$ for some nilpotent orbit. $\mathbb{O} \subset \mathcal{N}$.

. The inclusion $\mathcal{V}(J_\lambda) \subset \mathcal{N}$ is not so hard to prove

proof: $\chi_\lambda : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$ $\quad \ker \chi_\lambda \subset J_\lambda$ (if $z \in \ker \chi_\lambda$ $z \cdot m = \underbrace{\chi_\lambda(z)}_{=0} m$ $\forall m \in L_{\mathfrak{g}}(\lambda)$)

$\ker \chi_\lambda \subset Z(\mathfrak{g})$ maximal ideal of $Z(\mathfrak{g})$

. $J_\lambda \supset \ker \chi_\lambda$ and so $\mathrm{gr}\, J_\lambda$ contains all $\sigma(z)$ for $z \in Z(\mathfrak{g})$, where

$\sigma : Z(\mathfrak{g}) \longrightarrow \mathrm{gr}\, Z(\mathfrak{g}) = S(\mathfrak{g})^{\mathfrak{g}}$

then we can show that $\mathrm{gr}\, J_\lambda$ contains $S(\mathfrak{g})^{\mathfrak{g}}_+ \Rightarrow \mathcal{V}(J_\lambda) \subset \mathcal{V}(S(\mathfrak{g})^{\mathfrak{g}}_+) = \mathcal{N}$ $\square$.

Natural question: for $\lambda \in \mathfrak{h}^*$, what is $\mathbb{O}$ where $\mathcal{V}(J_\lambda) = \overline{\mathbb{O}}$?)

Prop: $\dim L_{\mathfrak{g}}(\lambda) < \infty$ $(\Leftrightarrow \lambda \in P^+)$ $\Leftrightarrow$ $\dim \mathcal{V}(J_\lambda) = \{0\}$ ie $\mathbb{O} = \{0\}$

proof "$\Rightarrow$" $\lambda \in P^+$.

. θ highest positive root of $\mathfrak{g}$

there exists $k \in \mathbb{Z}_{\geq 0}$ s.t $e_\theta^k \cdot m = 0$ $\forall m \in L_{\mathfrak{g}}(\lambda)$ $\Rightarrow e_\theta \in \sqrt{J_\lambda}$.

$e_\theta \in \mathfrak{g}_\theta \setminus \{0\}$.

(idea $L_{\mathfrak{g}}(\lambda)$ is generated by elements of $U(\mathfrak{n}_-)v_\lambda$ $\quad U(\mathfrak{g}) = \underline{U(\mathfrak{n}_-)} \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$

$e_{-\alpha_{i_1}}^{i_1} \cdots e_{-\alpha_r}^{i_r} v_\lambda$ $\quad \dim L_{\mathfrak{g}}(\lambda) < \infty$.)

- J_λ is $(\mathrm{ad}\,\mathfrak{g})$-invariant: if $u \in J_\lambda$, $x \in \mathfrak{g}$ $\quad (\mathrm{ad}\,x)u \in J_\lambda$

ad x acts by derivation on $U(\mathfrak{g})$ $\quad ((\mathrm{ad}\,x)(uv) = (\mathrm{ad}\,x)u\,v + u(\mathrm{ad}\,x)v)$

then $\sqrt{J_\lambda}$ is also $(\mathrm{ad}\,\mathfrak{g})$-invariant by the below lemma.

lemma: (R, ∂) diff alg, $I \subset R$ diff ideal. then $\partial\sqrt{I} \subset \sqrt{I}$
ie $\partial I \subset I$

- e_θ generates $\mathfrak{g}$ as $\mathfrak{g}$-module: $\exists$ chains $\beta_1 \dots \beta_n$ of positive roots s.t.
$\beta_n = \theta$, $\exists\, i \in \{1..\ell\}$, $\beta_{j+1} = \beta_j - \alpha_i$

- $e_\theta \in \sqrt{J_\lambda} \Rightarrow \sqrt{J_\lambda} \supset \mathfrak{g}$: $\quad \Gamma(J_\lambda) = \{0\}$ $\quad \boxtimes$

$\longrightarrow X_{L_k(\mathfrak{g})}$ $k \in \mathbb{Z}_{\geq 0}$. ??